Conservation of energy:

$$\frac{de}{dt}=\frac{dw}{dt}+\frac{dq}{dt}. \tag{3.3}$$

Here $\rho\equiv 1/v$ is material density, u is material velocity, e is specific internal energy, dw/dt is the rate at which work is done on unit mass, dq/dt is the rate at which heat is delivered to unit mass, d/dt denotes the convective derivative, $n=1$, 2 and 3 for plane, cylindrical and spherical waves, respectively.

If the convective derivative of entropy is small immediately behind the shock front, equations (3.2) and (3.3) are redundant. Consider this case first and suppose that

$$p_x=p_x(v,\xi) \tag{3.4}$$

where ξ is an additional physical variable on which p_x depends. It might, for example, be plastic strain, strain rate or electric field. Then

$$\frac{dp_x}{dt}=\frac{\partial p_x}{\partial v}\frac{dv}{dt}+\frac{\partial p_x}{\partial \xi}\frac{d\xi}{dt}=a^2\frac{d\rho}{dt}+\alpha\frac{d\xi}{dt} \tag{3.5}$$

where a is frozen sound speed, i.e. sound speed with $\xi=$ constant. Elimination of $d\rho/dt$ between equations (3.1) and (3.5) gives

$$\frac{dp_x}{dt}+a^2\rho\frac{\partial u}{\partial x}-\alpha\frac{d\xi}{dt}+\frac{\rho u a^2(n-1)}{x}=0. \tag{3.6}$$

Denote path of the shock front by $x=X(t)$ and shock velocity by $R=DX/Dt$. Derivative of any field variable $f(x,t)$ along a path parallel to the shock front is denoted Df/Dt:

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+R\frac{\partial f}{\partial x}=\frac{df}{dt}+(R-u)\frac{\partial f}{\partial x} \tag{3.7}$$

since $df/dt=\partial f/\partial t+u\partial f/\partial x$. Substitution of equation (3.7) into equations (3.2) and (3.6) gives the following pair:

$$\frac{Du}{Dt}-(R-u)\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p_x}{\partial x}-\frac{2(n-1)\tau}{\rho x} \tag{3.8}$$

$$\frac{Dp_x}{Dt}+a^2\rho\frac{\partial u}{\partial x}=(R-u)\frac{\partial p_x}{\partial x}+\alpha\frac{d\xi}{dt}-\frac{\rho u a^2(n-1)}{x}. \tag{3.9}$$

Now apply equations (3.8) and (3.9) to the region just behind the discontinuity representing the shock. The shock jump condition which represents the equation of motion is

$$p_x=\rho_0 R u, \tag{3.10}$$

where pressure in the unshocked state is assumed to be negligible and ρ_0 denotes unshocked mass density. Any change in shock pressure p_x is accompanied by changes in R and u:

$$\frac{1}{p_x}\frac{Dp_x}{Dt}=\frac{1}{R}\frac{DR}{Dt}+\frac{1}{u}\frac{Du}{Dt} \tag{3.11}$$

$$=\frac{A}{R}\frac{Dp_x}{Dt}+\frac{B}{R}\frac{D\xi}{Dt}+\frac{1}{u}\frac{Du}{Dt} \tag{3.12}$$

where

$$A=\frac{\partial R}{\partial p_x},\qquad B=\frac{\partial R}{\partial \xi}. \tag{3.13}$$

Equation (3.12) can be used to eliminate $\mathrm{D}u/\mathrm{D}t$ from equation (3.8). The result is

$$\left(\frac{u}{p_x}-\frac{uA}{R}\right)\frac{\mathrm{D}p_x}{\mathrm{D}t}-(R-u)\frac{\partial u}{\partial x}=\frac{uB}{R}\frac{\mathrm{D}\xi}{\mathrm{D}t}-\frac{1}{\rho}\frac{\partial p_x}{\partial x}-\frac{2(n-1)\tau}{\rho X}. \tag{3.14}$$

It is now possible to eliminate $\partial u/\partial x$ between equations (3.9) and (3.14):

$$\left[(R-u)+\frac{a^2\rho u}{p_x}-\frac{a^2\rho A u}{R}\right]\frac{\mathrm{D}p_x}{\mathrm{D}t}=[(R-u)^2-a^2]\frac{\partial p_x}{\partial x}-\frac{2a^2(n-1)\tau}{X}-\frac{\rho a^2 u(R-u)(n-1)}{X}$$
$$+\alpha(R-u)\frac{\mathrm{d}\xi}{\mathrm{d}t}+\frac{a^2\rho B u}{R}\frac{\mathrm{D}\xi}{\mathrm{D}t}. \tag{3.15}$$

With $\mathrm{D}\xi/\mathrm{D}t=\mathrm{d}\xi/\mathrm{d}t+(R-u)\ \partial\xi/\partial x$, equation (3.15) becomes

$$\frac{\mathrm{D}p_x}{\mathrm{D}t}=M\frac{\partial p_x}{\partial x}+L\frac{\mathrm{d}\xi}{\mathrm{d}t}+N\frac{\partial \xi}{\partial x}-\frac{G}{X} \tag{3.16}$$

where

$$M=(R-u)[(R-u)^2-a^2]/Q,$$
$$L=[\alpha(R-u)^2+a^2\rho_0 Bu]/Q,$$
$$N=a^2\rho_0 Bu(R-u)/Q,$$
$$Q=(R-u)^2+a^2(1-\rho_0 Au),$$
$$G=a^2(n-1)(p_x+2\tau)(R-u)/Q.$$

If R and u depend only on p_x, $B\equiv\partial R/\partial\xi=0$, and $A=\mathrm{d}R/\mathrm{d}p_x=(1-\rho_0 R\ \mathrm{d}u/\mathrm{d}p_x)\rho_0 u$. Using the identity $\rho(R-u)=\rho_0 R$, we find

$$1-\rho_0 Au=\rho_0 R\ \mathrm{d}u/\mathrm{d}p_x. \tag{3.17}$$

Divide equation (3.16) by R to obtain $\mathrm{D}p_x/\mathrm{D}X$. Then with $\xi=\text{const.}$, $a^2=c^2$. Set $\tau=0$; use equation (3.10), the shock jump condition $\rho(R-u)=\rho_0 R$ and equation (3.17) in equation (3.16) and it reduces to the Harris relation, equation (2.2). The effect of finite strength, represented by τ, is to increase the rate of geometric attenuation.

It may happen that R is very insensitive to ξ, so that the coefficient N vanishes, but L is still sensible. Then Maxwell attenuation proportional to $\mathrm{d}\xi/\mathrm{d}t$ will exist.

Examples

(i) *Elastic–plastic solids.* In an elastic–plastic-relaxing solid, outside the yield surface, p_x depends on both v and plastic strain, $\epsilon_x{}^p$. If stresses are supported by elastic strains alone, and if plastic dilatation vanishes[10],

$$\dot{p}_x=a^2\dot{\rho}-2\mu\dot{\epsilon}_x{}^p$$
$$\equiv a^2\dot{\rho}-F \tag{3.18}$$

where a^2 is independent of $\epsilon_x{}^p$ and F is the relaxation function. With $\xi=\epsilon_x{}^p$, $\alpha=-2\mu$, and $B=0$, equation (3.16) becomes

$$\frac{\mathrm{D}p_x}{\mathrm{D}t}=M\frac{\partial p_x}{\partial x}-\frac{2\mu(R-u)^2\dot{\epsilon}_x{}^p}{(R-u)^2+a^2(1-\rho_0 Au)}. \tag{3.19}$$

(ii) *Piezoelectric solids.* In an axial mode piezoelectric device a plane shock is made to propagate in the direction of polarization and a depolarization current, I, is produced in an external circuit. If p_x is allowed to depend on both ρ and electric displacement, D, ξ in equation